Orthogonality Relations for Characters

$\varphi : G \to GL(V)$ $\rightsquigarrow$ $(\chi_\varphi : G \to \mathbb{C}) \in L^c(G)$

<u>representation</u> <u>character</u>

$\chi_\varphi(g) = \mathrm{Tr}(\varphi_g)$

$f_1, f_2 \in L^c(G) \leq L(G)$:

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)} \in \mathbb{C}$$

inner product

<u>**Theorem**</u>: φ, ψ <u>irreducible</u> G-reps.

Then $\langle \chi_\varphi, \chi_\psi \rangle = \begin{cases} 1 & \varphi \sim \psi \\ 0 & \varphi \nsim \psi \end{cases}$

$\Longrightarrow$ "irreducible characters" form an orthonormal subset of $L^c(G)$

Cor: There are at most s equivalence classes of irreducible G-reps, where $s =$ number of conjugacy classes in G

Proof: φ, ψ irred, $\chi_\varphi = \chi_\psi \Longrightarrow \varphi \sim \psi$

$$\langle \chi_\varphi, \chi_\psi \rangle = \underset{\varphi\sim\varphi}{\langle \chi_\varphi, \chi_\varphi \rangle} = 1 \quad \text{Thm.}$$

$$\left\{\begin{array}{c}\text{irreducible reps} \\ \text{up to equivalence}\end{array}\right\} \xleftrightarrow{\text{bijection}} \{\chi_\varphi \mid \varphi \text{ irred}\} \subseteq L^c(G)$$

$$|\text{o.n. subset}| \leq \dim L^c(G) = s$$

Prop: $\varphi = \varphi^{(1)} \oplus \cdots \oplus \varphi^{(r)}$, $\varphi^{(k)}$ irred.

if $\lambda =$ irred G-rep.

$\langle \chi_\varphi, \chi_\lambda \rangle =$ number of indices $k \in \{1, \dots, n\}$ such that $\lambda \sim \varphi^{(k)}$

Proof: $\chi_\varphi = \chi_{\varphi^{(1)}} + \cdots + \chi_{\varphi^{(r)}}$

$$\langle \chi_\varphi, \chi_\lambda \rangle = \sum_{k=1}^{r} \underbrace{\langle \chi_{\varphi^{(k)}}, \chi_\lambda \rangle}_{1 \text{ or } 0, \text{ depending on } \varphi^{(k)} \sim \lambda}$$

Cor: Any two $\oplus$ decompositions of φ into irreducibles have the same number of irreducibles of each type

Write $\lambda^1, \ldots, \lambda^r$ for a complete list of pairwise inequivalent irreducible G-reps

Cor: If $\varphi : G \to GL(V)$, then

$$\underline{\varphi \sim m_1 \lambda^1 \oplus \cdots \oplus m_r \lambda^r} \quad \text{where}$$

$$m_k := \langle \chi_\varphi, \chi_{\lambda^k} \rangle \in \mathbb{Z}_{\geq 0} \quad \text{and}$$

$$\text{"} m\lambda \text{"} = \text{"} \underbrace{\lambda \oplus \cdots \oplus \lambda}_{m} \text{"}$$

$m_k =$ "multiplicities"

Proof: φ is completely reducible

$\exists$ equivalence: $\varphi \sim m_1 \lambda^1 \oplus \cdots \oplus m_r \lambda^r$, $m_k \in \mathbb{Z}_{\geq 0}$

$$\langle \chi_\varphi, \chi_{\lambda^k} \rangle = \left\langle \sum_{i=1}^{r} m_i \chi_{\lambda^i}, \chi_{\lambda^k} \right\rangle$$

$$= \sum_{i=1}^{r} m_i \underbrace{\cancel{\langle \chi_{\lambda^i}, \chi_{\lambda^k} \rangle}}_{}^{\delta_{ik}} = m_k.$$

Cor: φ, ψ representations of G

Then $\varphi \sim \psi$ iff $\chi_\varphi = \chi_\psi$
$\Longleftrightarrow$

Cor: $\varphi \sim m_1 \lambda^1 \oplus \cdots \oplus m_r \lambda^r$, $\lambda^1, \ldots, \lambda^r$ complete list of irred.

Then $\langle \chi_\varphi, \chi_\varphi \rangle = \sum_{k=1}^{r} m_k^2$

$\Longrightarrow$ $\langle \chi_\varphi, \chi_\varphi \rangle = 1$ iff φ irreducible.

Proof:

$$\langle \chi_\varphi, \chi_\varphi \rangle = \left\langle \sum_{i=1}^{r} m_i \chi_{\lambda^i}, \sum_{j=1}^{r} m_j \chi_{\lambda^j} \right\rangle$$

$$= \sum_{i,j} m_i m_j \langle \cancelto{\delta_{ij}}{\chi_{\lambda^i}, \chi_{\lambda^j}} \rangle$$

$$= \sum_i m_i^2$$

$\sum m_i^2 = 1$ iff $\begin{cases} m_k = 1 & \text{for some } k. \\ m_i = 0 & \text{if } i \neq k. \end{cases}$

$\Rightarrow \varphi \sim \lambda^k$.